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Comments on minimal interactions in quantum mechanics

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Abstract. We study minimal interactions in quantum systems characterized by position and momentum operators defined as the direct product of a traceless finite matrix and an ordinary canonical coordinate.

1. Introduction

There exists a large variety of quantum mechanical systems described by a complete set of *compatible* (non-interferring) observables which include, as an obvious additional entity, a multiple of a *reducible* identity matrix. Consider, for instance, the system described by the Jaynes–Cummings model in quantum optics which consists of a two-level atom interacting with a cavity mode [1–3], the well known supersymmetric systems in $(1+1)$ dimensions [4–8], and the familiar ones described by the Pauli wave equation. A common feature of these systems is that their Hilbert spaces are acted upon by an identity matrix of the form $I_{2\times 2}\otimes I$, where I is an infinite-dimensional identity matrix. There exists a non-trivial way of representing the position and momentum observables of the corresponding systems by making them formally 'traceless' operators. This property permits the introduction of a large number of new minimal interactions into the corresponding free particle wave equations. To be specific, let us consider a general quantum system described by canonical coordinates Q_i and P_j satisfying the Heisenberg algebra

$$[Q_i, P_j] = \mathrm{i}\hbar\mathbb{I}\delta_{ij} \tag{1.1}$$

where $\mathbb{I} \equiv I_{n\times n}\otimes I$ represents a n-block identity matrix such that we may realize these operators in the general form

$$Q_i = \hat{\eta}\otimes q_i \qquad P_j = \hat{\eta}\otimes p_j \tag{1.2}$$

where $p_j = -\mathrm{i}\hbar\partial/\partial q_j$ and $\hat{\eta}$ is a constant $n\times n$ Hermitian matrix operator satisfying $\hat{\eta}^2 = I_{n\times n}$. From equation (1.2) we can define a *label* Δ associated with each representation of the Heisenberg algebra (1.1)

$$n \geqslant \Delta(Q_i, P_j) \equiv |\,\mathrm{Tr}\,\hat{\eta}| \geqslant 0. \tag{1.3}$$

Representations satisfying $\Delta = n$ correspond to the usual ones ($\hat{\eta} = I_{n\times n}$) where Q_i, P_j are reducible operators for $n\geqslant 2$. In this paper we shall be concerned with representations for which $\Delta = 0$, i.e. n is an even integer and Q_i, P_j here are formally 'traceless' operators.

The Hilbert space is abstractly defined as

$$\mathcal{H} = L^2(\mathbb{R}^3)\otimes\mathbb{C}^n. \tag{1.4}$$

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It consists of n-component column vectors

$$\Psi(\boldsymbol{q}, t) = \begin{pmatrix} \psi_1(\boldsymbol{q}, t) \\ \vdots \\ \psi_n(\boldsymbol{q}, t) \end{pmatrix} \tag{1.5}$$

where each component ψ_i is a complex valued function of the *four*-dimensional (flat) spacetime coordinates $\boldsymbol{q}, t$. The scalar product is given by

$$(\Psi, \Phi) \equiv \int_{\mathbb{V}\subset\mathbb{R}^3} \Psi^\dagger(\boldsymbol{q}, t)\Phi(\boldsymbol{q}, t)\,\mathrm{d}^3q = \int_{\mathbb{V}\subset\mathbb{R}^3} \sum_{i=1}^{n} \psi_i^*(\boldsymbol{q}, t)\phi_i(\boldsymbol{q}, t)\,\mathrm{d}^3q. \tag{1.6}$$

The operator $\boldsymbol{Q}$ consists of three self-adjoint operators Q_i whose domains are defined as

$$\mathcal{D}(Q_i) = \left\{\Psi \in \mathcal{H}| \int_{\mathbb{V}\subset\mathbb{R}^3} (Q_i\Psi)^\dagger Q_i\Psi\,\mathrm{d}^3q = \int_{\mathbb{V}\subset\mathbb{R}^3} \sum_{j=1}^{n} |q_i\psi_j|^2\,\mathrm{d}^3q < \infty\right\}. \tag{1.7}$$

The momentum operator $P_j = -\mathrm{i}\hbar\hat{\eta} \otimes \partial/\partial q_j$ can be defined as the Fourier transformation of the position operator Q_j $(j = 1, 2, 3)$.

Minimal interactions can now be introduced by means of the prescription $P_\mu \to P_\mu - gA_\mu$, where g is the coupling constant, A_μ is a gauge field $(\mu = 0, 1, 2, 3)$. Note that here $P_0 = -\mathrm{i}\hbar I_{2\times 2} \otimes \partial/\partial q_0$. This is the basis of the so-called *gauge principle* whereby the form of the interaction is determined on the basis of local gauge invariance. The covariant derivative $D_\mu \equiv (\mathrm{i}/\hbar)(P_\mu - gA_\mu)$ turns out to be of fundamental importance to determine the field strength tensor of the theory. It will be the operator which generalizes from electromagnetic-like interactions.

In section 2 we introduce the interaction of a quantum system as above with an electromagnetic-like field specified by A_μ by taking over the procedure of minimal substitution. We then briefly examine the problem of gauge invariance of the theory. In section 3 we consider the *one*-dimensional problem in the non-relativistic limit. Two cases are discussed: a simple oscillator-like interaction and the interaction of a two-level atom with a two-mode electromagnetic field in a configuration consisting of two counterpropagating travelling waves.

Section 4 deals with the non-relativistic three-dimensional problem for the cases of a spinless and a spin-$\frac{1}{2}$ particle in a central vector potential. Finally, section 5 considers the relativistic problem for a Dirac particle. We study again the case of a spin-$\frac{1}{2}$ particle in a central vector potential taking as an example a Coulomb-like interaction. We also briefly discuss the *Zitterbewegung* of the free electron.

2. Minimal interactions

In what follows we shall omit the symbol '$\otimes$' when its presence is obvious as in equation (1.2). Let us consider a quantum system described by the free-particle Hamiltonian $H_0 = H_0(P_j)$. We can incorporate a minimal interaction into H_0 by making the substitution

$$P_\mu \to \Pi_\mu = P_\mu - gA_\mu \tag{2.1}$$

where $cP_0 = I_{n\times n} \otimes \mathrm{i}\hbar\partial/\partial t$ as usual, A_μ $(\mu = 0, 1, 2, 3)$ is a 4-vector potential and g is the corresponding coupling constant. Here

$$A_\mu(q) = \tau_a A_\mu^a(q) \tag{2.2}$$

with the $A_\mu^a(q)$ Hermitian functions of $q = (ct, q_j)$, and $\tau_a = (I_{n\times n}, \tau_r)$, where the τ_r are $n^2 - 1$ independent $n \times n$ traceless Hermitian matrices. Note that in equation (2.2) there is a summation on the index $a = 1, \ldots, n^2$. The system will now be described by the Schrödinger wave equation

$$H_0(\mathbf{\Pi})\Psi = (\mathrm{i}\hbar\partial_0 - g\tau_a A_0^a(q))\Psi \tag{2.3}$$

where $\Pi_i = P_i - gA_i$ $(i = 1, 2, 3)$ and the wavefunction Ψ is itself a n-component spinor. The minimal replacement (2.1) must also satisfy the (formal) Hermiticity condition

$$H_0^\dagger(\mathbf{\Pi}) = H_0(\mathbf{\Pi}) \tag{2.4}$$

which in general restricts the form of the vector field (2.2) (see, for instance, section 5).

The field strength is defined by the commutators between the components of the covariant derivative $D_\mu \equiv (\mathrm{i}/\hbar)\Pi_\mu$. By using equations (2.1) and (2.2) we find

$$\begin{aligned}
\frac{\mathrm{i}g}{\hbar}F_{ij} &\equiv [D_i, D_j] = \frac{\mathrm{i}g}{\hbar}\left\{\hat{\eta}(\partial_j A_i - \partial_i A_j) + [A_j, \hat{\eta}]\partial_i - [A_i, \hat{\eta}]\partial_j + \frac{\mathrm{i}g}{\hbar}[A_i, A_j]\right\} \\
\frac{\mathrm{i}g}{\hbar}F_{i0} &\equiv [D_i, D_0] = \frac{\mathrm{i}g}{\hbar}\left\{-\hat{\eta}(\partial_i A_0) + [A_0, \hat{\eta}]\partial_i - \partial_0 A_i + \frac{\mathrm{i}g}{\hbar}[A_0, A_i]\right\}.
\end{aligned} \tag{2.5}$$

If we want to keep the field strength components F_{ij} and F_{i0} antisymmetric and symmetric under spatial inversion $\boldsymbol{q} \to -\boldsymbol{q}$, we must demand the components of the gauge potential A_0 and A_i to be (up to a constant multiple of the identity matrix) symmetric and antisymmetric fields, respectively. Note that equations (2.1)–(2.5) are in direct correspondence to the usual Abelian case (an *electromagnetic* interaction) and not to a new non-Abelian generalization. The non-Abelian case follows straightforwardly *mutatis mutandis* as in the ordinary case and, to be brief, we shall not treat it here.

The solution of the wave equation (2.3) describes completely the state of the particle moving under the influence of the potential $A_\mu(q)$. This wave equation can be made gauge invariant under the combined local gauge transformation

$$\begin{aligned}
A_\mu(q) &\to A'_\mu(q) = A_\mu(q) + \delta A_\mu(q) \\
\Psi(q) &\to \Psi'(q) = \Psi(q) + \delta\Psi(q).
\end{aligned} \tag{2.6}$$

In the above we consider an infinitesimal local phase transformation for $\Psi(q)$ of the form

$$\Psi'(q) = U(q)\Psi(q) \tag{2.7}$$

where

$$U(q) = \mathbb{I} + \mathrm{i}g\tau_a\zeta^a(q) + \mathrm{O}(\zeta^2). \tag{2.8}$$

The requirement is that

$$D'_\mu\Psi'(q) = U(q)D_\mu\Psi(q)$$

with

$$D'_\mu = \hat{\eta}\partial_\mu - \frac{\mathrm{i}g}{\hbar}(A_\mu(q) + \delta A_\mu(q)). \tag{2.9}$$

Equation (2.9) holds true if

$$\begin{aligned}
\delta\Psi(q) &= \mathrm{i}g\tau_a\zeta^a(q)\Psi(q) \\
\delta A_0(q) &= \tau_a\partial_0\zeta^a(q) + \mathrm{i}g[\tau_a\zeta^a(q), A_0(q)] \\
\delta A_k(q) &= \hat{\eta}\tau_a\partial_k\zeta^a(q) + \zeta^a(q)[\tau_a, \hat{\eta}]\partial_k + \mathrm{i}g[\tau_a\zeta^a(q), A_k(q)]
\end{aligned} \tag{2.10}$$

which corresponds to a direct generalization of a usual local Abelian gauge transformation.

3. The one-dimensional case

A *one*-dimensional system is characterized by canonical coordinates Q and P satisfying

$$[Q, P] = \mathrm{i}\hbar\mathbb{I}. \tag{3.1}$$

For $n = 2$ we may represent these operators in the general form

$$Q = \hat{\eta}q \qquad P = \hat{\eta}p \qquad \mathbb{I} = I_{2\times2} \otimes I \tag{3.2}$$

where $p = -\mathrm{i}\hbar\partial/\partial q$, $\mathrm{Tr}\,\hat{\eta} = 0$ and $\hat{\eta}^2 = I_{2\times2}$. The free-particle Schrödinger equation can be written as

$$H_0\Psi = \frac{1}{2m}P^2\Psi = \frac{1}{2m}p^2\Psi = \mathrm{i}\hbar\partial_t\Psi. \tag{3.3}$$

As one would expect, the Hamiltonian H still describes a Schrödinger 'free particle', but Ψ is a two-component wavefunction now.

In the presence of an *electromagnetic* coupling, equation (2.3) reads ($g \equiv e/c$, $e < 0$)

$$H\Psi = \left(\frac{1}{2m}\left(P - \frac{e}{c}A_1\right)^2 + eA_0\right)\Psi = \mathrm{i}\hbar\partial_t\Psi \tag{3.4}$$

where

$$A_\mu = \tau_a A_\mu^a(q) \tag{3.5}$$

with $\mu = 0, 1$. Here we require the representation to have as low a rank as possible. We choose

$$\hat{\eta} \equiv \tau_1 \qquad \tau_a = (I_{2\times2}, \tau_i) \tag{3.6}$$

where the τ_i are the Pauli matrices. Thus the Schrödinger equation becomes

$$H\Psi = \left(\frac{1}{2m}p^2 - \frac{e}{2mc}\left(-\mathrm{i}\hbar\hat{\eta}\frac{\mathrm{d}A_1(q)}{\mathrm{d}q} + \{\hat{\eta}, A_1(q)\}p\right) + \frac{e^2}{2mc^2}A_1^2(q) + eA_0\right)\Psi$$
$$= \mathrm{i}\hbar\partial_t\Psi. \tag{3.7}$$

3.1. The harmonic oscillator-like case

Let us choose a particular representation satisfying

$$A_0 = 0 \qquad \{\tau_3, \Pi\} = 0 \tag{3.8}$$

where $\{\,,\}$ denotes the anticommutator and $\Pi = P - (e/c)A_1$. In this case the most general gauge potential will have the form

$$A_1(q) = U_1(q)\tau_1 + U_2(q)\tau_2 \tag{3.9}$$

where U_1, U_2 are general differentiable functions of q. Thus equation (3.7) reduces to

$$H\Psi = \left(\frac{1}{2m}p^2 + \mathrm{i}\hbar\frac{e}{2mc}\left(\frac{\mathrm{d}U_1(q)}{\mathrm{d}q} + \mathrm{i}\tau_3\frac{\mathrm{d}U_2(q)}{\mathrm{d}q}\right) - \frac{e}{mc}U_1(q)p\right.$$
$$\left. + \frac{e^2}{2mc^2}(U_1^2(q) + U_2^2(q))\right)\Psi = \mathrm{i}\hbar\partial_t\Psi. \tag{3.10}$$

Note that in this problem the set $\{\mathcal{H}, H, \tau_3, \Pi\}$ defines a supersymmetric system [7]. Here $i(\Pi) = \mathrm{Tr}[(\tau_3 \otimes I)\exp(-\beta H)]$ (well defined independently of $\beta > 0$ if $\exp(-\beta H)$ is of trace class) is an index which measures supersymmetry breaking [7].

To be specific, we set $A_0 = 0$, $U_2(q) = \lambda q$ (λ a constant), and $U_1(q)$ a general function of q. Equation (3.10) becomes

$$H\Psi = \left(\frac{1}{2m}\left(p - \frac{e}{c}U_1(q)\right)^2 - \frac{e\hbar\lambda}{2mc}\tau_3 + \frac{e^2\lambda^2}{2mc^2}q^2\right)\Psi = \mathrm{i}\hbar\partial_t\Psi \tag{3.11}$$

which corresponds to a harmonic oscillator system. By defining

$$a \equiv \frac{1}{\sqrt{2m\hbar\lambda}}\left(\lambda q + \mathrm{i}\left(p - \frac{e}{c}U_1(q)\right)\right) \tag{3.12}$$

with

$$[a, a^\dagger] = I \tag{3.13}$$

we get

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \tag{3.14}$$

where the supersymmetric partner Hamiltonians are given by

$$H_\mp = \frac{e\hbar\lambda}{mc}(a^\dagger a + \tfrac{1}{2}) \mp \frac{e\hbar\lambda}{2mc}. \tag{3.15}$$

The energy spectrum is then

$$E_n^+ = \frac{e\hbar\lambda}{mc}(n+1) \qquad E_n^- = \frac{e\hbar\lambda}{mc}n \tag{3.16}$$

for $n = 0, 1, \ldots$. Thus the ground state $E_{n=0}^-$ has zero energy and is non-degenerate. All excited states are doubly degenerate. This shows that supersymmetry in this system is unbroken, see also [7].

3.2. Atom–field interaction in a cavity

As a second instance, let us consider the interaction of a two-level atom in a configuration consisting of two counterpropagating travelling waves, for instance in a ring configuration [9, 10]. If the atom is assumed to propagate with momentum p_z in the z direction perpendicular to the light field, the atom–field Hamiltonian before the atom gets into the cavity is

$$\begin{aligned} H_0 &= H_K^{(\mathrm{Atom})} + V^{(\mathrm{Atom})} + H_K^{(\mathrm{Field})} \\ &= \frac{1}{2m}P^2 + \frac{\hbar\omega_0}{2}\tau_3 + \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathbb{I}) \end{aligned} \tag{3.17}$$

where we have chosen $P = -\mathrm{i}\hbar\hat{\eta}\partial/\partial q$, with q the direction of the light field, $\hat{\eta} = \tau_3$ and we have absorbed the kinetic energy of the atom associated with the z direction in the definition of H_0. In the above ω_0, ω are the atomic transition and field frequencies, respectively. In what follows we assume that the two frequencies are close to resonance, $\omega \sim \omega_0$. Once the atom is inside the cavity, the interaction can be incorporated through the minimal replacement

$$P_\mu \to \Pi_\mu = P_\mu - \frac{e}{c}A_\mu \tag{3.18}$$

with (effective) gauge potential

$$\begin{aligned} A_0 &= 0 \\ A_1 &= \left(\frac{\hbar c^2}{V\omega}\right)^{1/2}\{(a_2\tau_+ - a_1^\dagger\tau_-)\exp(\mathrm{i}kq) + (a_2^\dagger\tau_- - a_1\tau_+)\exp(-\mathrm{i}kq)\} \end{aligned} \tag{3.19}$$

where τ_-, τ_+ are the pseudo-spin lowering and raising operators, respectively. From equations (3.17)–(3.19) we get

$$H = \frac{1}{2m}\Pi_1^2 + \tfrac{1}{2}\hbar\omega_0\tau_3 + \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathbb{I})$$
$$= \frac{1}{2m}P^2 + \hbar\omega'(a_1^\dagger a_1 + a_2^\dagger a_2 + \mathbb{I}) + \tfrac{1}{2}\hbar\omega_0'\tau_3$$
$$+\hbar\Omega\{(a_1^\dagger\tau_- + a_2\tau_+)\exp(\mathrm{i}kq) + (a_1\tau_+ + a_2^\dagger\tau_-)\exp(-\mathrm{i}kq)\}$$
$$-\hbar\delta\omega\{a_1^\dagger a_2\exp(-2\mathrm{i}kq) + a_1 a_2^\dagger\exp(2\mathrm{i}kq)\} \qquad (3.20)$$

where

$$\omega' = \omega + \delta\omega \qquad \omega_0' = \omega_0 + 2\delta\omega \qquad \delta\omega = |e\mu|\frac{c}{\hbar V\omega} \qquad (3.21)$$

and

$$\Omega = |\mu|\left(\frac{\omega}{\hbar V}\right)^{1/2} \qquad (3.22)$$

with $\mu = \hbar e/2mc$. Note that in equation (3.22) we have obtained the correct value for the coupling constant Ω being a factor $\sqrt{2}$ larger than the one given for standing waves [11]. The expression for the Hamiltonian (3.20) differs from the one given by Shore, Meystre and Stenholm (SMS) [10]: the gauge potential (3.19) introduces a shift $\delta\omega$ in the field frequency, and the detuning becomes $\omega' - \omega_0' = \omega - \omega_0 - \delta\omega$. Furthermore, apart from the usual one-photon exchange interaction with coupling constant $\hbar\Omega$, there is a new (momentum conserving) *exchange mode* contribution in H with coupling constant $\hbar\delta\omega$. Note that for high enough field frequencies (visible optical frequencies) $\delta\omega$ becomes small. For instance, by choosing $\omega \sim 10^{11}$ Hz and $V = 1$ cm^3, one gets $\delta\omega/\Omega \sim 0.01$. Thus, in the limit of high field frequencies, the SMS result is re-obtained.

Using the fact that

$$\hat{M} \equiv a_1^\dagger a_1 + a_2^\dagger a_2 + \mathbb{I} + \tfrac{1}{2}\tau_3 \qquad \hat{N} \equiv -\frac{\mathrm{i}}{k}\frac{\mathrm{d}}{\mathrm{d}q} + a_1^\dagger a_1 - a_2^\dagger a_2 \qquad (3.23)$$

commute with each other and with the total Hamiltonian H, the basis states can be labelled as

$$|M, N; m^{(\tau)}, \tau\rangle \equiv \exp\left(\frac{\mathrm{i}}{\hbar}pq\right)|m_1^{(\tau)}\rangle|m_2^{(\tau)}\rangle|\tau\rangle \qquad (3.24)$$

with

$$M = m_1^{(\tau)} + m_2^{(\tau)} + 1 + \tfrac{1}{2}\tau \qquad \tau = \pm 1$$
$$N = p + \hbar k(m_1^{(\tau)} - m_2^{(\tau)}) \qquad (3.25)$$

where we have chosen $m^{(\tau)} \equiv m_1^{(\tau)} = 0, 1, \ldots, M+1-\tau/2$. Note that H involves generators of the superalgebra $u(2/1)$ generated by the bilinear products

$$S_i^{(+)} = \tau_- a_i^\dagger \qquad G_i^j = a_i a_j^\dagger \qquad S_i^{(-)} = \tau_+ a_i \qquad \tau_\pm\tau_\mp = \tfrac{1}{2}(I_{2\times 2} \pm \tau_3). \qquad (3.26)$$

Here the exponential factors $\exp(\pm\mathrm{i}kq)$ are implicit in the field operators. In fact $\hat{M}$ displays an $su(2)$ symmetry (subalgebra) whose generators are

$$J_+ = a_1^\dagger a_2 + \tau_+ \qquad J_- = a_2^\dagger a_1 + \tau_- \qquad J_3 = a_1^\dagger a_1 - a_2^\dagger a_2 + \tfrac{1}{2}\tau_3 \quad (3.27)$$

with

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k. \qquad (3.28)$$

We find that

$$[J_i, \hat{M}] = [J_i, \hat{N}] = 0 \tag{3.29}$$

being (3.24) a basis of representation for the elements of this algebra.

To determine the energy spectrum we note that $\Pi_1 = P - (e/c)A_1$ shares the same symmetry as H:

$$[\Pi_1, \hat{M}] = [\Pi_1, \hat{N}] = 0. \tag{3.30}$$

By using this fact we calculate the matrix elements of Π_1 in the basis (3.24) for given values of M, N. After squaring the corresponding $2M \times 2M$ diagonal matrix, we replace it in equation (3.20) yielding the exact spectrum of H. With the definition (3.24) we find

$$\begin{aligned}
&(M, N; m_a^{(\tau_a)}, \tau_a|\Pi_1|M, N; m_b^{(\tau_b)}, \tau_b) \\
&\quad = \hbar\tau(M + N - 2m_b^{(\tau_b)} - \tfrac{1}{2}(\tau_b + 2))k\delta_{m_a^{(\tau_a)},m_b^{(\tau_b)}}\delta_{\tau_a,\tau_b} \\
&\qquad + \frac{ef}{c}\bigg(\sqrt{m_b^{(\tau_b)}}\,\delta_{m_a^{(\tau_a)},m_b^{(\tau_b)}-1}\delta_{\tau_a,\tau_b+2} \\
&\qquad - \sqrt{M - m_b^{(\tau_b)} - \tfrac{1}{2}(\tau_b + 2)}\,\delta_{m_a^{(\tau_a)},m_b^{(\tau_b)}}\delta_{\tau_a,\tau_b+2} \\
&\qquad + \sqrt{m_b^{(\tau_b)} + 1}\,\delta_{m_a^{(\tau_a)},m_b^{(\tau_b)}+1}\delta_{\tau_a,\tau_b-2} \\
&\qquad - \sqrt{M - m_b^{(\tau_b)} - \tfrac{1}{2}\tau_b}\,\delta_{m_a^{(\tau_a)},m_b^{(\tau_b)}}\delta_{\tau_a,\tau_b-2}\bigg).
\end{aligned} \tag{3.31}$$

In figure 1, we show the energy spectra of the interacting Hamiltonian H for a rubidium atom ($m = 1.4192 \times 10^{-22}$ g) in units of $\hbar\omega$, $\omega = 21\,506.5$ MHz ($63p_{3/2}$–$61d_{3/2}$ transition), for the particular case $N = 0$. On the left we display the levels of $\hat{M}$ which are $2M$ degenerate. The energy levels of H_0 depicted on the right by full lines are, excluding the lowest one within each subspace, doubly degenerate. After the interaction is switched on, only the excited states within each subspace are shifted as shown in the same figure.

4. Three dimensions

4.1. A spinless non-relativistic particle in a Coulomb field

We first consider the Hamiltonian H for a spinless particle moving in an electromagnetic field A_μ

$$H = \frac{1}{2m}\left(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}\right)^2 + eA_0 \tag{4.1}$$

where $P_i = -\mathrm{i}\hbar\hat{\eta}\partial/\partial q_i$ and $A_i = \tau_a A_i^a$, with τ_a Pauli matrices and A_i^0, A_i^1, A_i^3, A_0 vanishing fields. Here $\hat{\eta} \equiv \tau_1$, while A_i^2 is a central vector field of the form

$$A_i^2(q) = U(q)\frac{q^i}{q} \tag{4.2}$$

with $U(q)$ a real function of $q = |\boldsymbol{q}|$. We obtain

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \tau_3\frac{\hbar e}{2mcq^2}\frac{\mathrm{d}}{\mathrm{d}q}(q^2U(q)) + \frac{e^2}{2mc^2}U^2(q). \tag{4.3}$$

It is clear that this Hamiltonian is not supersymmetric since the term proportional to $\mathrm{d}/\mathrm{d}q()$ involves a function which is different from U.

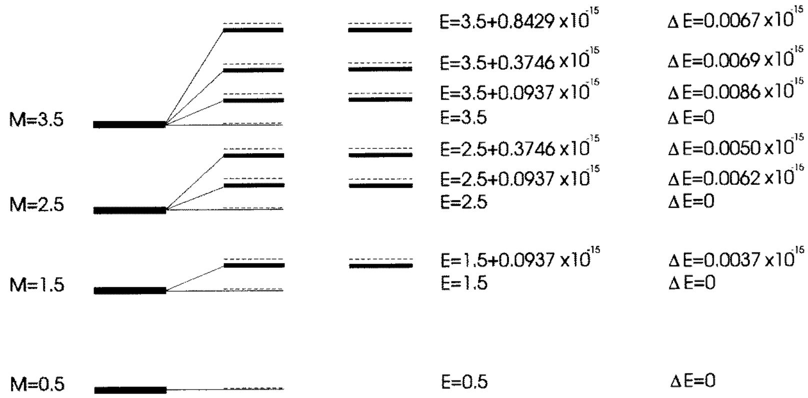

Figure 1. Schematic representation of the $2M$ degenerate energy levels for $N = 0$ in units of $\hbar\omega$ and their splitting after the atom–field interaction is turned on. Notice that, except for the lowest energy levels, there is a residual double degeneracy in the excited levels within each subspace.

By choosing $U = mZe/\hbar$, the Schrödinger equation for stationary states leads to the following two equations:

$$\left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{\hbar^2}{2m}l(l+1)\frac{1}{q^2} \mp \frac{Ze^2}{q} + \frac{Z^2e^4m}{2\hbar^2}\right\}\psi_\mp = E_\mp\psi_\mp. \tag{4.4}$$

Thus the Hilbert space $\{\Psi = (\Psi_+, \Psi_-)\}$ with

$$\Psi_+ = \begin{pmatrix}\psi_+\\ 0\end{pmatrix} \qquad \Psi_- = \begin{pmatrix}0\\ \psi_-\end{pmatrix} \tag{4.5}$$

consists of the complete set of bound and scattering states of the problem. Note that there is a double degeneracy for the continuum solutions: the eigenvalues E ($E > Z^2e^4m/2\hbar^2$) of (4.4) are those satisfying $E = E_+ = E_-$, with $\Psi_\pm$ the corresponding well known scattering solutions for the scalar potentials $\pm Ze^2/q$. The discrete solutions are described by Ψ_+ with energy eigenvalues

$$E_n = \frac{Z^2e^4m}{2\hbar^2}\left(1 - \frac{1}{n^2}\right) \tag{4.6}$$

with $n = 1, 2, \ldots$. Because H is proportional to the square of $\boldsymbol{P} - (e/c)\boldsymbol{A}$ we see that $E_n \geqslant 0$.

4.2. A spin-$\frac{1}{2}$ non-relativistic particle

The squared momentum of a spin-$\frac{1}{2}$ free particle may be written in the form

$$\boldsymbol{P}^2 = \left(\sum_{k=1}^{3} P_k \otimes \sigma_k\right)^2. \tag{4.7}$$

Thus the free-particle wave equation reads

$$H\Psi = \frac{1}{2m}\boldsymbol{P}^2\Psi = \frac{1}{2m}\left(\sum_k P_k \otimes \sigma_k\right)^2\Psi = \mathrm{i}\hbar\partial_t\Psi \tag{4.8}$$

with $P_k = \hat{\eta} p_k$. Under minimal interaction the Hamiltonian becomes

$$\begin{aligned} H &= \frac{1}{2m}\left(\left(P_k - \frac{e}{c}A_k\right)\otimes\sigma_k\right)^2 + eA_0 \\ &= \frac{1}{2m}\left(P_k - \frac{e}{c}A_k\right)^2 - \mathrm{i}\frac{e}{mc}\varepsilon_{ijk}[P_i, A_j]\otimes \tfrac{1}{2}\sigma_k + eA_0 \end{aligned} \tag{4.9}$$

where there is summation over the indices i, j, k. In equation (4.9) we have chosen the same two-dimensional matrix structure for P_μ, A_μ as in the previous case. From equation (2.5) the *magnetic* field is defined by

$$B_k \equiv \tfrac{1}{2}\varepsilon_{ijk}F_{ij} = \frac{\mathrm{i}c}{2\hbar e}\varepsilon_{ijk}[\Pi_i, \Pi_j] = -\frac{\mathrm{i}}{\hbar}\varepsilon_{ijk}[P_i, A_j]. \tag{4.10}$$

Thus the Hamiltonian can be written as

$$\begin{aligned} H &= \frac{1}{2m}\left(\left(P_k - \frac{e}{c}A_k\right)\otimes\sigma_k\right)^2 + eA_0 \\ &= \frac{1}{2m}\left(P_k - \frac{e}{c}A_k\right)^2 + \frac{\boldsymbol{g}e}{2mc}B_k\otimes S_k + eA_0 \end{aligned} \tag{4.11}$$

where $S_k \equiv \frac{1}{2}\sigma_k$, and $\boldsymbol{g} = 2$ is the gyromagnetic constant. Hence this Hamiltonian represents the interaction of a magnetic moment $(\hbar e/2mc)\sigma$ with the *magnetic* field $\boldsymbol{B}$ felt in the rest frame of the particle.

As an example let us consider again a central potential of the form $A_\mu = (0, \tau_2 U(q)\hat{\boldsymbol{q}})$. The Schrödinger equation for the system becomes

$$\begin{aligned} H\Psi &= \left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{e^2}{2mc^2}U^2(q) - \frac{\hbar e}{2mc}\tau_3\otimes\left(\frac{\mathrm{d}U(q)}{\mathrm{d}q} + \frac{2U(q)}{q}\left(\mathbb{I} + \frac{1}{\hbar}\boldsymbol{L}\cdot\sigma\right)\right)\right)\Psi \\ &= \mathrm{i}\hbar\partial_t\Psi \end{aligned} \tag{4.12}$$

where now the four-component spinor wavefunction Ψ has the form

$$\Psi = \begin{pmatrix}\Phi_+ \\ \Phi_-\end{pmatrix}. \tag{4.13}$$

Note that the Hamiltonian H in equation (4.12) is supersymmetric [12]. Hence, under a particular (vector-like) minimal interaction, H turns out to be supersymmetric only if $\boldsymbol{g} = 2$. The supersymmetric partners $V_\pm$ are in this case

$$V_\pm(q) = \frac{e^2}{2m}U^2(q) \pm \frac{\hbar e}{2m}\left(\frac{\mathrm{d}U(q)}{\mathrm{d}q} + \frac{2U(q)}{q}\left(\mathbb{I} + \frac{1}{\hbar}\boldsymbol{L}\cdot\sigma\right)\right). \tag{4.14}$$

According to equation (2.5) the *magnetic* field is defined by

$$\begin{aligned} B_k &\equiv F_{ij} = [A_j, \hat{\eta}]\partial_i - [A_i, \hat{\eta}]\partial_j \\ &= -\frac{2U(q)}{q}\tau_3 L_k \end{aligned} \tag{4.15}$$

$(i, j, k$ cyclic), i.e. it is proportional to the angular momentum of the particle.

The standard spin–orbit coupling in quantum mechanics due to an electric potential $V(q)$ takes the form

$$H_{QM}^{(LS)} = \frac{\hbar e}{4m^2c^2 q}\frac{\mathrm{d}V(q)}{\mathrm{d}q}\boldsymbol{L}\cdot\sigma. \tag{4.16}$$

In contrast with this, the spin–orbit coupling required by (4.12) is

$$H^{(LS)} = \frac{e}{m}\frac{U(q)}{q}\boldsymbol{L}\cdot\sigma. \tag{4.17}$$

If we now look for a *direct* identification

$$U(q) \to \frac{\hbar}{4mc^2}\frac{\mathrm{d}V(q)}{\mathrm{d}q} \qquad U^2(q) \to \frac{2mc^2}{e}V(q) \tag{4.18}$$

we get $U(q) = (4m^2c^3/\hbar e)q$. This system corresponds to a harmonic oscillator with frequency $\omega = 4mc^2/\hbar$. This result can be interpreted by saying that the harmonic oscillator would be supersymmetric if the ground-state energy is sufficient to create a pair [12, 13].

Notice that the spin–orbit coupling introduces a correction to the result given in (4.4) for the *Coulomb* field,

$$-\frac{Ze^2}{q} \to -\frac{\hbar e}{2m}\left(\frac{2mZe/\hbar}{q}\left(\mathbb{I} + \frac{1}{\hbar}\boldsymbol{L}\cdot\sigma\right)\right) = -\frac{Ze^2}{q}\left(\mathbb{I} + \frac{1}{\hbar}\boldsymbol{L}\cdot\sigma\right) \tag{4.19}$$

associated to the Φ_+ wavefunction component. Thus the resulting effective potential will strongly depend on the values of the orbital and total angular momenta j, l of the state system (see section 5 for the solution of the corresponding relativistic problem).

5. The relativistic case for a spin-$\frac{1}{2}$ particle

Our considerations can also be extended to relativistic quantum mechanics. In the case of a Dirac particle we already count with a 4×4 (formally) *traceless* Hamiltonian. Here it seems natural to also look for appropriate 4×4 (formally) *traceless* momentum and coordinate operators. To this end we notice that the Dirac wave equation can be written as

$$H\Psi = (c\boldsymbol{\Sigma}\cdot\boldsymbol{P} + mc^2\beta)\Psi = \mathrm{i}\hbar\partial_t\Psi \tag{5.1}$$

where

$$P_k \equiv -\gamma_5 p_k = \mathrm{i}\hbar\gamma_5\nabla_k \qquad \Sigma_k = -\gamma_5\alpha_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{5.2}$$

with

$$\gamma_5 = -\begin{pmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix}. \tag{5.3}$$

Thus we define

$$\begin{aligned} Q_0 &\equiv I_{4\times 4}\otimes q_0 \qquad & P_0 &\equiv I_{4\times 4}\otimes p_0 \\ Q_i &\equiv -\gamma_5\otimes q_i \qquad & P_j &\equiv -\gamma_5\otimes p_j \end{aligned} \tag{5.4}$$

where $p_\mu = \mathrm{i}\hbar\partial/\partial q^\mu$. The operators Q_μ, P_ν satisfy the canonical commutation relations

$$[Q_\mu, Q_\nu] = [P_\mu, P_\nu] = 0 \qquad [Q_\mu, P_\nu] = \mathrm{i}\hbar g_{\mu\nu}\mathbb{I} \tag{5.5}$$

where $\mathbb{I} = I_{4\times 4}\otimes I$ and $g = \mathrm{diag}(-1, 1, 1, 1)$.

Next we introduce the minimal replacement

$$P_\mu \to P_\mu - \frac{e}{c}A_\mu \tag{5.6}$$

with

$$A_\mu(q) = \Gamma_a A_\mu^a(q) \qquad A_\mu^\dagger = A_\mu \tag{5.7}$$

where the matrices Γ_a belong to the complete set of 16 *Gamma* matrices

$$I_{4\times 4} \qquad \mathrm{i}\gamma_\mu \qquad \mathrm{i}\gamma_\mu\gamma_\upsilon(\upsilon \neq \mu) \qquad \gamma_5 \qquad \mathrm{i}\gamma_5\gamma_\mu. \tag{5.8}$$

By replacing equations (5.6) and (5.7) in equation (5.1) we get

$$H\Psi = \left(c\Sigma_k\left(P_k - \frac{e}{c}A_k(q)\right) + mc^2\beta\right)\Psi = (\mathrm{i}\hbar\partial_t - eA_0(q))\Psi. \tag{5.9}$$

Note that if we demand (formally) H to be an Hermitian operator we find that

$$[\Gamma_a, \Sigma_k] A^a_k(q) = 0 \tag{5.10}$$

which is a constraint on the form of the gauge potential A_k.

Equation (5.9) may be re-written as

$$\left(\Upsilon^\mu D_\mu + \frac{mc}{\hbar}\right)\Psi = 0 \tag{5.11}$$

where

$$\Upsilon^0 = \beta \qquad \Upsilon^k = \beta\Sigma_k. \tag{5.12}$$

Note also that, apart from the basic condition (2.9), gauge invariance will require

$$[\Upsilon^\mu, U(q))] = 0 \tag{5.13}$$

which is a strong restriction on the space of allowed gauge transformations for a Dirac particle. From (5.13) we find that the most general gauge transformation U has the form

$$U(q) = \exp \mathrm{i} g(\zeta^0(q) + \zeta^1(q)\beta). \tag{5.14}$$

On the basis of the equations (5.6)–(5.10) a variety of problems can be considered. As an example, let us choose

$$A_0(q) = 0 \qquad A_k(q) = \mathrm{i}\beta\gamma_5 \frac{mZe}{\hbar}\frac{q_k}{q}. \tag{5.15}$$

By writing

$$\Psi = \begin{pmatrix}\Phi_+ \\ \Phi_-\end{pmatrix} \tag{5.16}$$

we see that for steady states equation (5.9) consists of two coupled equations

$$c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} \mp \mathrm{i}\frac{mZe^2}{\hbar}\frac{\boldsymbol{q}}{q}\right)\Phi_\pm = (E \pm mc^2)\Phi_\mp. \tag{5.17}$$

After decoupling them we find

$$\left(\frac{1}{2m}\boldsymbol{p}^2 \mp \frac{Ze^2}{q}\left(I + \frac{1}{\hbar}\boldsymbol{\sigma}\cdot\boldsymbol{L}\right) + \frac{Z^2e^4m}{2\hbar^2}\right)\Phi_\pm = (2mc^2)^{-1}(E^2 - m^2c^4)\Phi_\pm. \tag{5.18}$$

This may as well be written in the compact form

$$\left(\frac{1}{2m}\boldsymbol{p}^2 - \frac{Ze^2}{q}K\right)\Psi = (\epsilon - \epsilon_0)\Psi \tag{5.19}$$

with

$$\epsilon = \frac{E^2}{2mc^2} \qquad \epsilon_0 = \tfrac{1}{2}mc^2(1 + Z^2\alpha^2) \tag{5.20}$$

and

$$K = \beta\left(I + \frac{1}{\hbar}\boldsymbol{\Sigma}\cdot\boldsymbol{L}\right) = \frac{1}{\hbar}\beta\boldsymbol{\Sigma}\cdot\boldsymbol{J} - \beta/2 \qquad \boldsymbol{J} \equiv \boldsymbol{L} + \frac{\hbar}{2}\boldsymbol{\Sigma}. \tag{5.21}$$

Consider the non-relativistic limit. For positive energy solutions the norm of Φ_- is small in this limit and the wave equation reduces to (5.18) for Φ_+. Then $\epsilon - \epsilon_0$ is the eigenvalue of the steady states of (4.12) with the corresponding potential given in (4.19).

By using the fact that

$$\boldsymbol{L}\times\boldsymbol{q} + \boldsymbol{q}\times\boldsymbol{L} = 2\mathrm{i}\hbar\boldsymbol{q} \tag{5.22}$$

we find that K is a constant of motion

$$[H, K] = 0. \tag{5.23}$$

In addition to this we have that

$$[\boldsymbol{J}, K] = 0. \tag{5.24}$$

Thus we can construct simultaneous eigenfunctions of H, $\boldsymbol{J}^2$, K, J_3 just as in the case of the relativistic hydrogen atom [14]. From equation (5.21) we get

$$K^2 = \boldsymbol{J}^2 + \tfrac{1}{4} \tag{5.25}$$

so the eigenvalues of K are

$$\kappa = \pm(j + \tfrac{1}{2}). \tag{5.26}$$

The bound states are characterized by $\kappa > 0$. In this case the wavefunctions $\Phi_\pm$ are eigenfunctions of $\boldsymbol{L}^2$ with eigenvalues $l_\pm(l_\pm + 1)$, $l_\pm = j \pm \frac{1}{2}$. The solution of (5.18) is then given by

$$\begin{aligned}
\Phi_+(\boldsymbol{q}) &= \langle \boldsymbol{q}|+, n' j j_3\rangle = \sum_{m_+, s} c(j j_3; l_+ m_+, \tfrac{1}{2} s) q_{n', l_+}(q) Y_{l_+ m_+}(\theta, \varphi)\chi_s \\
\Phi_-(\boldsymbol{q}) &= \langle \boldsymbol{q}|-, n' j j_3\rangle = \sum_{m_-, s} c(j j_3; l_- m_-, \tfrac{1}{2} s) q_{n'-1, l_-}(q) Y_{l_- m_-}(\theta, \varphi)\chi_s.
\end{aligned} \tag{5.27}$$

Here $c(\ldots)$ are Clebsch–Gordan coefficients, $q_{n', l_\pm}$ are radial wavefunctions of the non-relativistic hydrogen atom, $Y_{l_\pm m_\pm}(\theta, \varphi)$ are spherical harmonics and χ_s correspond to the spinors

$$\chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.28}$$

From (5.19) the energy eigenvalues are given by

$$E_{n', \kappa} = mc^2 \left(1 + (Z\alpha)^2 \left(1 - \frac{|\kappa|^2}{(n' + |\kappa|)^2}\right)\right)^{1/2} \qquad n' = 0, 1, 2 \ldots. \tag{5.29}$$

The degeneracy of the energy levels is infinity: the levels $E_{n'_i, \kappa_i}$, $i = 1, 2, 3, \ldots$, satisfying

$$\frac{|\kappa_1|^2}{(n'_1 + |\kappa_1|)^2} = \frac{|\kappa_2|^2}{(n'_2 + |\kappa_2|)^2} = \cdots \tag{5.30}$$

have the same energy.

It is worth mentioning here that the Dirac oscillator [15] can also be re-obtained through a minimal interaction of the form (5.6)–(5.7) in the Dirac equation. This is done by choosing

$$A_k(q) = \mathrm{i}\beta\gamma_5 \frac{m\omega}{e} q_k \tag{5.31}$$

where ω is the frequency for this oscillator. This gauge field gives rise to a harmonic oscillator with a strong spin–orbit coupling which introduces, as in the previous case, an infinite degeneracy. This oscillator has a hidden supersymmetry, responsible for the special properties of its spectrum [16]. In fact we can easily see that by setting $U(q) = (mc\omega/e)q$ in equation (4.12) we re-obtain, up to a constant term in the Hamiltonian, the Dirac oscillator in the non-relativistic limit. It is interesting to note that the vector field A_k given in equation (5.31), is a Hermitian operator. This feature is absent in Moshinski's approach [15].

5.1. The electron Zitterbewegung

We briefly discuss the *Zitterbewegung* (trembling motion) of the Dirac electron in the representation (5.4). To this end let us consider a free Dirac particle described by equation (5.1). In the Heisenberg picture, the time derivative of an operator, say T, is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[H, T]. \tag{5.32}$$

We can easily see that both the coordinate and the momentum operators are *not* constants of the motion despite the fact that the particle is free. From (5.1), (5.4) and (5.32), the equations of motion for Q_k and P_k are

$$\frac{\mathrm{d}Q_k}{\mathrm{d}t} = \frac{2\mathrm{i}mc^2}{\hbar}\beta Q_k + c\Sigma_k \qquad \frac{\mathrm{d}P_k}{\mathrm{d}t} = \frac{2\mathrm{i}mc^2}{\hbar}\beta P_k \tag{5.33}$$

while from equations (5.32) and (5.33) we get

$$\frac{\mathrm{d}^2 Q_k}{\mathrm{d}t^2} = \frac{2\mathrm{i}}{\hbar}\left(H\frac{\mathrm{d}Q_k}{\mathrm{d}t} - c^2 P_k\right). \tag{5.34}$$

Let us define [17], for an observable G and for energy E

$$G_A \equiv \tfrac{1}{2}(G + HE^{-1}GHE^{-1}) \tag{5.35}$$

as the observable relative to which the *Zitterbewegung* takes place. By using (5.4) and (5.35) we get

$$\begin{aligned} \boldsymbol{\xi_Q} &\equiv \boldsymbol{Q} - \boldsymbol{Q}_A = mc^2 H^{-1}\beta\boldsymbol{Q} - \frac{\mathrm{i}\hbar c}{2}H^{-1}(\boldsymbol{\Sigma} - c\boldsymbol{P}H^{-1}) \\ \boldsymbol{\xi_P} &\equiv \boldsymbol{P} - \boldsymbol{P}_A = mc^2 H^{-1}\beta\boldsymbol{P} \end{aligned} \tag{5.36}$$

with

$$T_r(\xi_{\boldsymbol{Q}}) = T_r(\xi_{\boldsymbol{P}}) = \boldsymbol{0}. \tag{5.37}$$

This result strongly differs from the usual expressions [17]. In particular the (odd) position and momentum operators $\boldsymbol{Q}$, $\boldsymbol{P}$ are themselves present in the corresponding *Zitterbewegung* coordinates.

6. Final comments

In this paper, we have studied minimal interactions in a wide class of quantum systems characterized by position and momentum operators defined as the direct product of a (Hermitian and unitary) finite traceless matrix and an ordinary canonical coordinate. This approach allows us to obtain in a simple fashion supersymmetric systems in quantum mechanics. However, we are not restricted only to this class of systems as was shown in the examples given in sections 3.2 and 4.1. In fact, equations (2.1) and (2.2) also include ordinary minimal interactions. In section 2 we worked out in detail the bound states of a two-level atom interacting with a two-mode electromagnetic field in a particular familiar configuration. This example suggests a useful gauge approach to some problems in cavity QED. A promising natural development of the present work is its extension to relativistic quantum field theory. We hope to report on such an extension elsewhere.

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